Classical and Quantum Cosmology of an Accelerating Model Universe with Compactification of Extra Dimensions

F. Darabi[*]

Department of Physics, Azarbaijan University of Tarbiat Moallem, 53714-161, Tabriz, Iran .

Abstract

We study a $(4 + D)$-dimensional Kaluza-Klein cosmology with a Robertson-Walker type metric having two scale factors a and R, corresponding to D-dimensional internal space and 4-dimensional universe, respectively. By introducing an exotic matter in the form of perfect fluid with an special equation of state, as the space-time part of the higher dimensional energy-momentum tensor, a four dimensional effective decaying cosmological term appears as $\lambda \sim R^{-m}$ with $0 \leq m \leq 2$, playing the role of an evolving dark energy in the universe. By taking $m = 2$, which has some interesting implications in reconciling observations with inflationary models and is consistent with quantum tunneling, the resulting Einstein's field equations yield the exponential solutions for the scale factors a and R. These exponential behaviors may account for the dynamical compactification of extra dimensions and the accelerating expansion of the 4-dimensional universe in terms of Hubble parameter, H. The acceleration of the universe may be explained by the negative pressure of the exotic matter. It is shown that the rate of compactification of higher dimensions as well as expansion of 4-dimensional universe depends on the dimension, D. We then obtain the corresponding Wheeler-DeWitt equation and find the general exact solutions in D-dimensions. A good correspondence between the solutions of classical Einstein's equations and the solutions of quantum Wheeler-DeWitt equation in any dimension, D, is obtained based on Hartle's point of view concerning the classical limits of quantum cosmology.

[*]e-mail: f.darabi@azaruniv.edu

1 Introduction

Cosmological models with a cosmological term Λ are currently serious candidates to describe the dynamics of our four dimensional universe. The history of cosmological term dates back to Einstein, and its original role was to allow static homogeneous solutions to Einstein's equations in the presence of matter which turned out to be unnecessary when the expansion of the universe was discovered . However, particle physicists then realized that the non-vanishing cosmological constant can be interpreted as a measure of the energy density of the vacuum which turned out to be the sum of a number of apparently disjoint contributions of quantum fields. In fact, a dynamical characteristic for the vacuum energy density (cosmological term) was attributed by quantum field theorists since the developments in particle physics and inflationary scenarios. According to modern quantum field theory, the structure of a vacuum turns out to be interrelated with some spontaneous symmetry-breaking effects through the condensation of quantum (scalar) fields. This phenomenon gives rise to a non-vanishing vacuum energy density of the form $< T_{\mu\nu} >= - < \rho > g_{\mu\nu}$. Therefore, the observed (or effective) cosmological term receives an extra contribution from $< T_{\mu\nu} >$ as follows:

$$\Lambda = \lambda + 8\pi G < \rho >,$$

where λ is the bare cosmological constant and G is the gravitational constant. From quantum field theory we may expect $< \rho >\approx M_{Pl}^4 \approx 2 \times 10^{71} GeV^4$ (M_{Pl} is the Planck mass), or another energy scale related to some spontaneous symmetry breaking effect such as M_{SUSY}^4 or M_{Weak}^4. Therefore, the bare cosmological constant receives potential contributions from these mass scales resulting in a large effective cosmological term. However, the experimental upper bound on the present value of the cosmological term, Λ, provided by measurements of the Hubble constant, H, reads numerically as

$$\frac{|\Lambda|}{8\pi G} \leq 10^{-29} g/cm^3 \approx 10^{-47} GeV^4,$$

which is too far from the expectation of quantum field theory . The question of why the observed vacuum energy is so small in comparison to the scales of particle physics is known as the cosmological constant problem. It is generally thought to be easier to imagine an unknown mechanism which would set Λ exactly to zero than one which would suppress it by just the right amount to yield an observationally tiny cosmological constant. If Λ is a dynamical variable (or vacuum parameter), then it is natural to suppose that in an expanding universe the cosmological term relaxes to the present tiny value by some relaxation mechanism which may be provided by a time-varying vacuum with a rolling scalar field [2].

There are still other possibilities to be advocated. In recent years, several attempts in these directions have been done, in the context of quantum cosmology [3]. One plausible explanation for a tiny cosmological term is to suppose that Λ is dynamically evolving and not constant, i.e., $\Lambda \propto R^{-m}$, where R is the scale factor of the universe and m is a parameter. So, as the universe expands from its small size in the early universe, the initially large effective cosmological term evolves and reduces to its present small value[4].

The study of Λ-decaying cosmological models has recently been the subject of particular interest both from classical and quantum aspects. The Λ decaying models may serve as poten-

tial candidates to solve this problem by decaying the large value of the cosmological constant Λ to its present observed value.

Also, there are strong (astronomical) observational motivations for considering cosmological models in which Λ is dynamically decreasing as $\Lambda \propto R^{-m}$. Some models assume *a priori* a fixed value for the parameter m. The case $m = 2$, corresponding to the cosmic string matter, has mostly been taken based on dimensional considerations by some authors [5]. The case $m \approx 4$ which resembles the ordinary radiation has also been considered by some other authors [6]. A third group of authors have also studied the case $m = 3$ corresponding to the ordinary matter [7]. There are also some other models in which the value of m is not fixed *a priori* and the numerical bounds on the value of m is estimated by observational data or obtained by calculation of the quantum tunnelling rate [8]. Other aspects of Λ-decaying models have also been discussed with no specific numerical bounds on m [9]. It is clear that the functional dependence $\Lambda \propto R^{-m}$ is phenomenological and does not result from the first principles of particle physics. However, for some domain for example, $0 \leq m < 3$, the decaying law $\Lambda \propto R^{-m}$ deserves further investigation. One important reason is that the age of the universe, in these models, is always larger than the age obtained in the standard Einstein-de Sitter cosmology, or the one we get in an open universe. Therefore, if we are interested in solving the age problem, the decaying Λ term appears to be a good candidate. In fact, according to the ansatz $\Lambda \propto R^{-m}$, one may suppose the natural value $< \rho > \approx M_{Pl}^4$ to be the value of Λ at the Planck time when R was of the order of the Planck length. Theoretically this ansatz does not directly solve the cosmological constant problem, but it relates this problem to the age problem of why our universe is so old and have a radius R much larger than the Planck length. In other words, this ansatz reduces two above problems to one problem of "Why our universe could have escaped the death at the Planck time", which seems to be the most natural fate of a baby-universe in quantum cosmology? One may assume that the value of Λ in the early universe might have been much bigger than its present value and large enough to drive some symmetry breakings which might have occurred in the early universe.

On the other hand, the idea that our 4-dimensional universe might have emerged from a higher dimensional space-time is now receiving much attention [10] where the compactification of higher dimensions plays a key role. However, the question of how and why this compactification occurs remains as an open problem. From string theory we know that the compactification may take place provided that the higher dimensional manifold admits special properties, namely if the geometry of the manifold allows, for example, the existence of suitable Killing vectors. However, it is difficult to understand why such manifolds are preferred and whether other possible mechanisms for compactification do exist. In cosmology, on the other hand, different kinds of compactifications could be considered. For example, in an approach, called *dynamical compactification* , the extra dimensions evolve in time towards very small sizes and the extra-dimensional universe reduces to an effective four-dimensional one. This type of compactification was considered in the context of Modern Kaluza-Klein theories [18]. It is then a natural question that how an effective four dimensional universe evolve in time and whether the resulting cosmology is similar to the standard Friedmann-Robertson-Walker four dimensional universe without extra dimensions.

Meanwhile, the recent distance measurements of type Ia *supernova* suggest strongly an accelerating universe [11]. This accelerating expansion is generally believed to be driven by an

energy source called *dark energy* which provides negative pressure, such as a positive cosmological constant [12], or a slowly evolving real scalar field called *quintessence* [13]. Moreover, the basic conclusion from all previous observations that ~ 70 percent of the energy density of the universe is in a dark energy sector, has been confirmed after the recent WMAP [14].

To model a universe based on these considerations one may start from a fundamental theory including both gravity and standard model of particle physics. In this regards, it is interesting to begin with ten or eleven-dimensional space-time of superstring/M-theory, in which case one needs a compactification of ten or eleven-dimensional supergravity theory where an effective 4-dimensional cosmology undergoes acceleration. However, it has been known for some time that it is difficult to derive such a cosmology and has been considered that there is a no-go theorem that excludes such a possibility, if one takes the internal space to be time-independent and compact without boundary [15]. However, it has recently been shown that one may avoid this no-go theorem by giving up the condition of time-independence of the internal space; and a solution of the vacuum Einstein equations with compact hyperbolic internal space has been proposed based on this model [16]. Similar accelerating cosmologies can also be obtained for SM2 and SD2 branes , not only for hyperbolic but also for flat internal space [17].

On the other hand, from cosmological point of view, it is not so difficult to find cosmological models in which the 4-dimensional universe undergoes an accelerating expansion and the internal space contracts with time, exhibiting the *dynamical compactification* [18], [19], [20].

In [20], for instance, it is shown that using a more general metric, as compared to Ref.[18], and introducing matter without specifying its nature, the size of compact space evolves as an inverse power of the radius of the universe. The Friedmann-Robertson-Walker equations of the standard four-dimensional cosmology is obtained using an effective pressure expressed in terms of the components of the higher dimensional energy-momentum tensor, and the negative value of this pressure may explain the acceleration of our present universe.

To the author's knowledge the question of Λ-decaying cosmological model has not received much attention in higher dimensional Kaluza-Klein cosmologies. Moreover, the exotic matter has not been considered as an alternative candidate to produce the acceleration of the universe. The purpose of the present chapter is to study a $(4 + D)$-dimensional Kaluza-Klein cosmology ,with an extended Robertson-Walker type metric, in this context [1]. As we are concerned with cosmological solutions, which are intrinsically time dependent, we may suppose that the internal space is also time dependent. It is shown that by taking this higher dimensional metric and introducing a 4-dimensional exotic matter, a decaying cosmological term $\Lambda \sim R^{-m}$ with $0 \leq m \leq 2$ is appeared as a type of dark energy, and for the case $m = 2$ the resulting field equations yield the exponential solutions for the scale factors of the four-dimensional universe and the internal space. These solutions may account for the accelerating universe and dynamical compactification of extra dimensions, driven by the negative pressure of the exotic matter [1]. It should be noted, however, that the solutions in principle describe typical inflation rather than the recently observed acceleration of the universe which is known to take place in an ordinary matter dominated universe. Nevertheless, regarding the fact that about 70 percent of the total energy density of the universe is of dark energy type with negative

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[1]A similar work [21] has already been done in which the same extended FRW metric was chosen with a radiation fluid occupying all the extended space-time. They found an inflation for 3-dimensions and a contraction for the D remaining spatial dimensions.

pressure, we may approximate the matter content of the universe with almost dark energy and consider the present model as a rather simplified model of a real accelerating universe.

The quantum cosmology of this model is also studied by obtaining the Wheeler-DeWitt equation and finding its general exact solutions. It is then shown that a good correspondence exists between the classical and quantum cosmological solutions, based on the interpretation of Hartle of the classical limits of quantum cosmology.

The chapter is organized as follows: In section **2**, we introduce the classical cosmology model by taking a higher dimensional Robertson-Walker type metric and a higher dimensional matter whose non-zero part is a four-dimensional exotic matter. In section **3**, we obtain the Einstein equations for the two scale factors. In section **4**, we solve the Einstein equations and obtain the solutions. In section **5**, we study the corresponding quantum cosmology and derive the Wheeler-DeWitt equation. In section **6**, the exact solutions of the Wheeler-DeWitt equation is obtained. Finally, in section **7**, we show a good correspondence between the classical and quantum cosmology. The chapter is ended with concluding remarks.

2 Classical cosmology

To begin with, we study the metric considered in [22] in which the space-time is assumed to be of Robertson-Walker type having a (3+1)-dimensional space-time part and an internal space with dimension D. We adopt a real chart $\{t, r^i, \rho^a\}$ with t, r^i, and ρ^a denoting the time, space coordinates and internal space dimensions, respectively. We, therefore, take[2]

$$ds^2 = -N^2(t)dt^2 + R^2(t)\frac{dr^i dr^i}{(1 + \frac{kr^2}{4})^2} + a^2(t)\frac{d\rho^a d\rho^a}{(1 + k'\rho^2)}, \tag{1}$$

where $N(t)$ is the lapse function, $R(t)$ and $a(t)$ are the scale factor of the universe and the radius of internal space, respectively; $r^2 \equiv r^i r^i (i = 1, 2, 3)$, $\rho^2 \equiv \rho^a \rho^a (a = 1, ...D)$, and $k, k' = 0, \pm 1$, reflecting flat, open or closed type of four-dimensional universe and D-dimensional space. We assume the internal space to be flat with compact topology S^D, which means $k' = 0$. This assumption is motivated by the possibility of the compact spaces to be flat or hyperbolic in "*accelerating cosmologies from compactification*" scenarios, as discussed in Introduction.

The form of energy-momentum tensor is dictated by Einstein's equations and by the symmetries of the metric (1). Therefore, we may assume

$$T_{AB} = (-\rho, p, p, p, p_D, p_D, ..., p_D), \tag{2}$$

where A and B run over both the space-time coordinates and the internal space dimensions. Now, we examine the case for which the pressure along all the extra dimensions vanishes, namely $p_D = 0$. In so doing, we are motivated by the *brane world* scenarios where the matter is to be confined to the 4-dimensional universe, so that all components of T_{AB} is set to zero but the space-time components [23] and it means no matter escapes through the extra dimensions.

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[2]There is a little difference between this metric and that of [22], in that here the lapse function is generally considered as $N(t)$ instead of taking $N = 1$.

We assume the energy-momentum tensor $T_{\mu\nu}$ of space-time to be an exotic χ fluid with the equation of state

$$p_\chi = (\frac{m}{3} - 1)\rho_\chi, \tag{3}$$

where p_χ and ρ_χ are the pressure and density of the fluid, respectively and the parameter m is restricted to the range $0 \leq m \leq 2$ [24]. It is worth noting that the equation of state (3) with $0 \leq m \leq 2$ resembles a universe with negative pressure matter, violating the strong energy condition [25] and this violation is required for a universe to be accelerated [16][3].

Using standard techniques we obtain the scalar curvature corresponding to the metric (1)

$$\mathcal{R} = \frac{-6Ra N \ddot{R} + 6Ra \dot{N} \dot{R} - 2R^2 \ddot{a} N + 2R^2 \dot{N} \dot{a} - 2aN^3 k + aN^3 k^2 r^2 - 6aN\dot{R}^2 - 6R\dot{R}\dot{a}N}{R^2 N^3 a},$$

and then substitute it into the dimensionally extended Einstein-Hilbert action (without higher dimensional cosmological term) plus a matter term indicating the above mentioned exotic fluid. This leads to the effective Lagrangian [4]

$$L = \frac{1}{2N} Ra^D \dot{R}^2 + \frac{D(D-1)}{12N} R^3 a^{D-2} \dot{a}^2 + \frac{D}{2N} R^2 a^{D-1} \dot{R}\dot{a} - \frac{1}{2} kNRa^D + \frac{1}{6} N\rho_\chi R^3 a^D, \tag{4}$$

where a dot represents differentiation with respect to t. We now take a closed $(k = 1)$ universe. Although the flat universe $(k = 0)$ is almost favored by observations, we will show an equivalence between $(k = 1)$ and $(k = 0)$ universes. One may obtain the continuity equation by using the contracted Bianchi identity in $(4+D)$ dimensions, namely

$$\nabla_M G^{MN} = \nabla_M T^{MN} = 0,$$

together with the assumption that the matter is confined to $(3+1)$-dimensional space-time as

$$T_{ab} = T_{\mu a} = 0,$$

which gives rise to

$$\nabla_\mu T^{\mu\nu} = 0,$$

or

$$\dot{\rho}_\chi R + 3(p_\chi + \rho_\chi)\dot{R} = 0. \tag{5}$$

It is easily shown that substituting the equation of state (3) into the continuity equation (5) leads to the following behavior of the energy density in a closed $(k = 1)$ Friedmann-Robertson-Walker universe [24]

$$\rho_\chi(R) = \rho_\chi(R_0) \left(\frac{R_0}{R}\right)^m \tag{6}$$

where R_0 is the value of the scale factor at an arbitrary reference time t_0.

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[3]Given Einstein equations, this condition on the energy-momentum tensor implies a condition on Ricci tensor as $R_{00} \geq 0$.

[4]We take the Planck units, $G = c = \hbar = 1$

Now, if we believe that the cosmological term plays the role of vacuum energy density, we may define the cosmological term [8]

$$\Lambda \equiv \rho_\chi(R), \tag{7}$$

which leads to

$$L = \frac{1}{2N} R a^D \dot{R}^2 + \frac{D(D-1)}{12N} R^3 a^{D-2} \dot{a}^2 + \frac{D}{2N} R^2 a^{D-1} \dot{R} \dot{a} - \frac{1}{2} N R a^D + \frac{1}{6} N \Lambda R^3 a^D, \tag{8}$$

where the cosmological term is now decaying with the scale factor R as

$$\Lambda(R) = \Lambda(R_0) \left(\frac{R_0}{R} \right)^m . \tag{9}$$

Note that Λ is now playing the role of an evolving dark energy [26] in 4-dimensions, because we did not consider explicitly a $(4+D)$ dimensional cosmological term in the action, and Λ appears merely due to the specific choice of the equation of state (3) for the exotic matter. The decaying Λ term may also explain the smallness of the present value of the cosmological constant since as the universe evolves from its small to large sizes the large initial value of Λ decays to small values. This phenomenon may somehow alleviate the cosmological constant problem.

Of particular interest, to us, among the different values of m is $m = 2$ which has some interesting implications in reconciling observations with inflationary models [27], and is consistent with quantum tunnelling [8].

3   Einstein equations

We take $m = 2$ and set the initial values of R_0 and $\Lambda(R_0)$ as

$$\Lambda(R_0) R_0^2 = 3 \quad , \quad \Lambda(R) = \frac{3}{R^2}, \tag{10}$$

leading to a positive cosmological term which, according to (7), guarantees the weak energy condition $\rho_\chi > 0$.

The lapse function $N(t)$, in principle, is also an arbitrary function of time due to the fact that Einstein's general relativity is a reparametrization invariant theory. We, therefore, take the gauge

$$N(t) = R^3(t) a^D(t). \tag{11}$$

Now, the Lagrangian becomes

$$L = \frac{1}{2} \frac{\dot{R}^2}{R^2} + \frac{D(D-1)}{12} \frac{\dot{a}^2}{a^2} + \frac{D}{2} \frac{\dot{R} \dot{a}}{R a}, \tag{12}$$

where Eq.(10) has been used. It is seen that the parameters k and Λ are effectively removed from the Lagrangian and this implies that although k and Λ are not zero in this model the corresponding 4-dimensional universe is equivalent to a flat universe with a zero cosmological

term. In other words, we do not distinguish between our familiar 4-dimensional universe, which seems to be flat and without any exotic fluid, and a closed universe filled with an exotic fluid. We now define the new variables

$$X = \log R \quad , \quad Y = \log a. \tag{13}$$

The lagrangian (12) is written as

$$L = \frac{1}{2}\dot{X}^2 + \frac{D(D-1)}{12}\dot{Y}^2 + \frac{D}{2}\dot{X}\dot{Y}. \tag{14}$$

The equations of motion are obtained

$$\ddot{X} + \frac{D}{2}\ddot{Y} = 0, \tag{15}$$

$$\ddot{X} + \frac{D-1}{3}\ddot{Y} = 0. \tag{16}$$

Combining the equations (15) and (16) we obtain

$$\ddot{X} = 0, \tag{17}$$

$$\ddot{Y} = 0. \tag{18}$$

4 Solutions of Einstein equations

The solutions for X and Y in Eqs. (17) and (18) are obtained

$$X = \mathcal{A}t + \gamma, \tag{19}$$

$$Y = \mathcal{B}t + \delta, \tag{20}$$

and the solutions for $R(t)$ and $a(t)$ are then as follows

$$R(t) = Ae^{\alpha t}, \tag{21}$$

$$a(t) = Be^{\beta t}, \tag{22}$$

where the constants "$\mathcal{A}$, $\mathcal{B}$, γ and δ" or "A, B, α and β" should be obtained, in principle, in terms of the initial conditions. It is a reasonable assumption that the size of all spatial dimensions be the same at $t = 0$. Moreover, it may be assumed that this size would be the Planck size l_p in accordance with quantum cosmological considerations. Therefore, we take $R(0) = a(0) = l_P$ so that $A = B = l_p$, and

$$R(t) = l_p e^{\alpha t}, \tag{23}$$

$$a(t) = l_p e^{\beta t}. \tag{24}$$

It is important to note that the constants α, β are not independent, and a relation may be obtained between them. This is done by imposing the zero energy condition $H = 0$ which is the well-known result in cosmology due to the existence of arbitrary laps function $N(t)$ in the theory. The Hamiltonian constraint is obtained through the Legender transformation of the Lagrangian (14)

$$H = \frac{1}{2}\dot{X}^2 + \frac{D(D-1)}{12}\dot{Y}^2 + \frac{D}{2}\dot{X}\dot{Y} = 0, \tag{25}$$

which is written in terms of α and β as

$$H = \frac{1}{2}\alpha^2 + \frac{D(D-1)}{12}\beta^2 + \frac{D}{2}\alpha\beta = 0. \tag{26}$$

This constraint is satisfied only for $\alpha \leq 0, \beta \geq 0$ or $\alpha \geq 0, \beta \leq 0$.

For $D \neq 1$, the case $\alpha = 0$ or $\beta = 0$ gives rise to time independent scale factors, namely $R = a = l_P$, which is not physically viable since we know, at least based on observations, the scale factor of the universe is time dependent. We, therefore, choose $\alpha > 0, \beta < 0$ so that the universe and the internal space would expand and contract, respectively, in accordance with the present observations.

For the case $D = 1$, we find

$$\begin{cases} \beta = \text{arbitrary} \\ \alpha = 0 \end{cases} \quad \text{or} \quad \alpha = -\beta. \tag{27}$$

The former is not physically viable, since it predicts no time evolution for the universe. The latter, however, may predict exponential expansion for $R(t)$, and exponential contraction for $a(t)$, both with the same exponent $\alpha > 0$.

For the general case $D > 1$, we find

$$\alpha_\pm = \frac{D\beta}{2}\left[-1 \pm \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}\right], \tag{28}$$

which gives two positive values for α indicating two possible expanding universes provided $\beta < 0$ which indicates the compactification of extra dimensions. Moreover, the values of $\alpha_\pm$, for a given negative value of β, become larger for higher dimensions. Therefore, the universe expands more rapidly in both possibilities. On the contrary, for a given positive value of α, indicating an expanding universe, the parameter β may take two negative values

$$\beta_\pm = \frac{2\alpha}{D}\left[-1 \pm \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}\right]^{-1}, \tag{29}$$

indicating two ways of compactification. Moreover, they become smaller for higher dimensions, exhibiting lower rates of compactification.

To find the constants α, β we first obtain the Hubble parameter for $R(t)$

$$H = \frac{\dot{R}}{R} = \alpha, \tag{30}$$

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by which the constant α is fixed. The observed positive value of H will then justify our previous assumption, $\alpha > 0$. We may, therefore, write the solutions (23) and (24) in terms of the Hubble parameter H as

$$R(t) = l_p e^{Ht}, \tag{31}$$

$$a(t) = l_p e^{-Ht}, \tag{32}$$

for $D = 1$, and

$$R(t) = l_p e^{Ht}, \tag{33}$$

$$a(t)_\pm = l_p e^{\frac{2Ht}{D}\left[-1\pm\sqrt{1-\frac{2}{3}(1-\frac{1}{D})}\right]^{-1}}, \tag{34}$$

and

$$R_\pm(t) = l_p e^{\frac{D\beta t}{2}\left[-1\pm\sqrt{1-\frac{2}{3}(1-\frac{1}{D})}\right]}, \tag{35}$$

$$a(t) = l_p e^{\beta t}. \tag{36}$$

for $D > 1$.

For a given $H > 0$, it is seen that the solution corresponding to $D = 1$ may predict an accelerating (de Sitter) universe and a contracting internal space with exactly the same rates. For $D > 1$, in Eqs.(33) and (34), for a given $H > 0$ in the exponent of $R(t)$ the exponent in $a(t)$ takes two negative values and becomes smaller for higher dimensions. This means that while the 4-dimensional (de Sitter) universe is expanding by the rate H, the higher dimensions may be compactified in two possible ways with different rates of compactification as a function of dimension, D. In Eqs.(35) and (36), on the other hand, for a given $\beta < 0$ the exponent in $R(t)$ takes two positive values which become larger for higher dimensions. This also means that while the extra dimensions contract by the rate β, the universe may be expanded in two possible ways with different expansion rates as a function of D.

It is easy to show that the Lagrangian (14) (or the equations of motion) is invariant under the simultaneous transformation

$$R \to R^{-1} \quad , \quad a \to a^{-1}, \tag{37}$$

which is consistent with the time reversal $t \to -t$. Therefore, four different phases of "*expansion-contraction*" for $R(t)$ and $a(t)$ are distinguished, Eqs.(33) - (36). One may prefer the "*expanding $R(t)$ - contracting $a(t)$*" phase to "*expanding $a(t)$ - contracting $R(t)$*" one, considering the present status of the 4D universe [5].

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[5] For the special case $D = 3$, both the Lagrangian (14) and the Hamiltonian constraint (25) are invariant under the transformation

$$a \to R \quad , \quad R \to a.$$

Therefore, we have a dynamical symmetry between R and a, namely

$$a \leftrightarrow R.$$

In this case there is no real line of demarcation between a and R to single out one of them as the real scale factor of the universe. This is because the internal space is flat $k' = 0$ and according to (12) one may assume the 4D universe with $k, \Lambda \neq 0$ to be equivalent to the one in which $k = \Lambda = 0$. Therefore, both have the same topology S^3.

The deceleration parameter q for the scale factor R is obtained

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = -1. \tag{38}$$

Observational evidences not only do not rule out the negative deceleration parameter but also puts the limits on the present value of q as $-1 \le q < 0$ [11]. Therefore, this negative value seems to favor a cosmic acceleration in the expansion of the universe.

In the expansion phase of the closed ($k = 1$) universe the cosmological term Λ decays exponentially with time t as

$$\Lambda(t) = 3l_p^{-2}e^{-2Ht}, \tag{39}$$

whereas in the contraction phase ($t \to -t$) it grows exponentially to large values so that at $t = 0$ it becomes extremely large, of the order of M_p^2. This huge value of Λ may be extinguished rapidly by assuming a sufficiently large Hubble parameter H, consistent with the present observations, to alleviate the cosmological constant problem.

5   Quantum cosmology

An appropriate quantum mechanical description of the universe is likely to be afforded by quantum cosmology which was introduced and developed by DeWitt [28]. In quantum cosmology the universe, as a whole, is treated quantum mechanically and is described by a single wave function, $\Psi(h_{ij}, \phi)$, defined on a manifold (*superapace*) of all possible three geometries and all matter field configurations. The wave function $\Psi(h_{ij}, \phi)$ has no explicit time dependence due to the fact that there is no a real time parameter external to the universe. Therefore, there is no Schrödinger wave equation but the operator version of the Hamiltonian constraint of the Dirac canonical quantization procedure [29], namely vanishing of the variation of the Einstein-Hilbert action S with respect to the arbitrary lapse function N

$$H = \frac{\delta S}{\delta N} = 0,$$

which is written

$$\hat{H}\Psi(h_{ij}, \phi) = 0.$$

This equation is known as the Wheeler-DeWitt (WDW) equation. The goal of quantum cosmology by solving the WDW equation is to understand the origin and evolution of the universe, quantum mechanically. As a differential equation, the WDW equation has an infinite number of solutions. To get a unique viable solution , we should also respect the question of boundary condition in quantum cosmology which is of prime importance in obtaining the relevant solutions for the WDW equation.

In principle, it is very difficult to solve the WDW equation in the *superspace* due to the large number of degrees of freedom. In practice, one has to *freeze out* of all but a finite number of degrees of freedom of the gravitational and matter fields. This procedure is known as quantization in *minisuperspace*, and will be used in the following discussion.

The minisuperspace in our model is two-dimensional with gravitational variables X and Y. To obtain the Wheeler-DeWitt equation, in this minisuperspace, we start with the Lagrangian (14). The conjugate momenta corresponding to X and Y are obtained

$$P_X = \frac{\partial L}{\partial \dot{X}} = \dot{X} + \frac{D}{2}\dot{Y}, \tag{40}$$

$$P_Y = \frac{\partial L}{\partial \dot{Y}} = \frac{D}{2}\dot{X} + \frac{D(D-1)}{6}\dot{Y}, \tag{41}$$

from which we obtain

$$\dot{X} = \frac{6}{D+2}\left[P_X\left(\frac{1-D}{3}\right) + P_Y\right], \tag{42}$$

$$\dot{Y} = \frac{6}{D(D-1)}\left[P_Y\frac{2(1-D)}{D+2} - P_X\frac{D(1-D)}{D+2}\right]. \tag{43}$$

Substituting Eqs.(42), (43) into the Hamiltonian constraint (25), we obtain

$$H = (1-D)P_X^2 - \frac{6}{D}P_Y^2 + 6P_XP_Y = 0. \tag{44}$$

Now, we may use the following quantum mechanical replacements

$$P_X \to -i\frac{\partial}{\partial X} \quad , \quad P_Y \to -i\frac{\partial}{\partial Y},$$

by which the Wheeler-DeWitt equation is obtained

$$\left[(D-1)\frac{\partial^2}{\partial X^2} + \frac{6}{D}\frac{\partial^2}{\partial Y^2} - 6\frac{\partial}{\partial X}\frac{\partial}{\partial Y}\right]\Psi(X,Y) = 0, \tag{45}$$

where $\Psi(X,Y)$ is the wave function of the universe in the (X,Y) mini-superspace.

We introduce the following change of variables

$$x = X(1 - \frac{D}{D+3}) + \frac{D}{D+3}Y \quad , \quad y = \frac{X-Y}{D+3}, \tag{46}$$

by which the Wheeler-DeWitt equation takes a simple form

$$\left\{-3\frac{\partial^2}{\partial x^2} + \frac{D+2}{D}\frac{\partial^2}{\partial y^2}\right\}\Psi(x,y) = 0. \tag{47}$$

Now, we can separate the variables as $\Psi(x,y) = \phi(x)\psi(y)$ to obtain the following equations

$$\frac{\partial^2\phi(x)}{\partial x^2} = \frac{\gamma}{3}\phi(x), \tag{48}$$

$$\frac{\partial^2\psi(y)}{\partial y^2} = \frac{\gamma D}{D+2}\psi(y), \tag{49}$$

where we assume $\gamma > 0$.

6 Solutions of Wheeler-DeWitt equation

The solutions of Eqs.(48), (49) in terms of x, y are as follows

$$\phi(x) = e^{\pm\sqrt{\frac{7}{3}}x}, \tag{50}$$

$$\psi(y) = e^{\pm\sqrt{\frac{\gamma D}{D+2}}y}, \tag{51}$$

leading to the four possible solutions for $\Psi(x, y)$ as

$$\Psi_D^{\pm}(x, y) = A^{\pm} e^{\pm\sqrt{\frac{7}{3}}x \pm \sqrt{\frac{\gamma D}{D+2}}y}, \tag{52}$$

$$\Psi_D^{\pm}(x, y) = B^{\pm} e^{\pm\sqrt{\frac{7}{3}}x \mp \sqrt{\frac{\gamma D}{D+2}}y}, \tag{53}$$

or alternative solutions in terms of X, Y as

$$\Psi_D^{\pm}(x, y) = A^{\pm} e^{\pm\sqrt{\frac{\gamma}{3}}\left(\frac{3X+DY}{D+3}\right) \pm \sqrt{\frac{\gamma D}{D+2}}\left(\frac{X-Y}{D+3}\right)}, \tag{54}$$

$$\Psi_D^{\pm}(x, y) = B^{\pm} e^{\pm\sqrt{\frac{\gamma}{3}}\left(\frac{3X+DY}{D+3}\right) \mp \sqrt{\frac{\gamma D}{D+2}}\left(\frac{X-Y}{D+3}\right)}, \tag{55}$$

where $A^{\pm}, B^{\pm}$ are the normalization constants. We may also write down the solutions in terms of R and a [6]

$$\Psi_D^{\pm}(R, a) = A^{\pm} R^{\pm\frac{1}{D+3}\left(\sqrt{3\gamma}+\sqrt{\frac{\gamma D}{D+2}}\right)} a^{\pm\frac{1}{D+3}\left(\sqrt{\frac{\gamma}{3}}D-\sqrt{\frac{\gamma D}{D+2}}\right)}, \tag{56}$$

$$\Psi_D^{\pm}(R, a) = B^{\pm} R^{\pm\frac{1}{D+3}\left(\sqrt{3\gamma}-\sqrt{\frac{\gamma D}{D+2}}\right)} a^{\pm\frac{1}{D+3}\left(\sqrt{\frac{\gamma}{3}}D+\sqrt{\frac{\gamma D}{D+2}}\right)}. \tag{57}$$

It is now important to impose the *good* boundary conditions on the above solutions to single out the physical ones. In so doing, we may impose the following condition

$$\Psi_D(R \to \infty, a \to \infty) = 0, \tag{58}$$

which requires the wave function of the universe to be normalizable. This means that our minisuperspace model has no classical solutions that expand simultaneously to infinite values of a and R, as Eqs.(31)-(36) show. Then, one may take the following solutions

$$\Psi_D^{\pm}(R, a) = C^{\pm} R^{-\frac{1}{D+3}\left(\sqrt{3\gamma}\pm\sqrt{\frac{\gamma D}{D+2}}\right)} a^{-\frac{1}{D+3}\left(\sqrt{\frac{\gamma}{3}}D\mp\sqrt{\frac{\gamma D}{D+2}}\right)}, \tag{59}$$

where $C^{\pm}$ are the normalization constants and the exponents of R and a are negative for any value of D [7].

One may obtain the solutions (59) in (X, Y) mini-superspace as

$$\Psi_D^{\pm}(x, y) = C^{\pm} e^{-\sqrt{\frac{\gamma}{3}}\left(\frac{3X+DY}{D+3}\right) \mp \sqrt{\frac{\gamma D}{D+2}}\left(\frac{X-Y}{D+3}\right)}. \tag{60}$$

---

[6]For $D = 3$, there is a exchange symmetry $\Psi(R, a) \leftrightarrow \Psi(a, R)$ under the exchange $a \leftrightarrow R$.

[7]For $D = 1$, the exponent of "a" corresponding to Ψ^{+} becomes zero so that Ψ^{+} depends only on R with the condition $\Psi^{+}(R \to \infty) \to 0$.

7   Correspondence between classical and quantum cosmology

One of the most interesting topics in the context of quantum cosmology is the mechanisms through which the classical cosmology may emerge from quantum theory. When does a Wheeler-DeWitt wave function predict a classical space-time? Quantum cosmology is the quantum mechanics of an isolated system (universe). It is not possible to use the Copenhagen interpretation, which needs the existence of an external observer, since here the observer is part of the system. Indeed, any attempt in constructing a viable quantum gravity requires understanding the connections between classical and quantum physics. Much work has been done in this direction over the past decade. Actually, there is some tendency towards using semiclassical approximations in dividing the behaviour of the wave function into two types, oscillatory or exponential which are supposed to correspond to classically allowed or forbidden regions. Hartle [30] has put forward a simple rule for applying quantum mechanics to a single system (universe): *If the wave function is sufficiently peaked about some region in the configuration space we predict to observe a correlation between the observables which characterize this region.* Halliwell [31] has shown that the oscillatory semiclassical WKB wave function is peaked about a region of the *minisuperspace* in which the correlation between the coordinate and momentum holds good and stresses that both *correlation* and *decoherence* are necessary before one can say a system is classical. Using Wigner functions, Habib and Laflamme [32] have studied the mutual compatibility of these requirements and shown that some form of coarse graining is necessary for classical prediction from WKB wave functions. Alternatively, Gaussian or coherent states with sharply peaked wave functions are often used to obtain classical limits by constructing wave packets.

In the investigation of classical limits, we first take $D = 1$ and look for a correspondence between classical and quantum solutions. Using Eqs.(31) and (32) in the Planck units, the corresponding classical locus in (R, a) configuration space, is

$$Ra = 1, \tag{61}$$

whereas in (X, Y) coordinates we have

$$X + Y = 0. \tag{62}$$

We now consider the wave functions (60) in (X, Y) mini-superspace for $D = 1$

$$\Psi_1^+(X, Y) = C^+ e^{-\sqrt{\frac{\gamma}{3}}X}, \tag{63}$$

$$\Psi_1^-(X, Y) = C^- e^{-\sqrt{\frac{\gamma}{3}}\frac{X+Y}{2}}. \tag{64}$$

The above wave functions, in their present form, are not square integrable as is required for the wave functions to predict the classical limit. However, one may take the absolute value of the exponents to make the wave functions square integrable

$$\Psi_1^+(X, Y) = C^+ e^{-|\sqrt{\frac{\gamma}{3}}X|}, \tag{65}$$

$$\Psi_1^-(X, Y) = C^- e^{-|\sqrt{\frac{2}{3}}\frac{X+Y}{2}|}. \tag{66}$$

We next consider the general case $D > 1$. Eliminating the parameter t in Eqs. (33) and (34) the classical loci in terms of R, a are obtained

$$a_\pm = R^{\frac{2}{D}\left[-1\pm\sqrt{1-\frac{2}{3}(1-\frac{1}{D})}\right]^{-1}}. \tag{67}$$

The corresponding forms of these loci in terms of X, Y are

$$Y_+ = \frac{2}{D}X\left[-1 + \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}\right]^{-1}, \tag{68}$$

$$Y_- = \frac{2}{D}X\left[-1 - \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}\right]^{-1}. \tag{69}$$

The wave functions (60) also are not square integrable, so we may replace the exponents by their absolute values

$$\Psi_D^\pm(x, y) = C^\pm e^{-\left|\sqrt{\frac{2}{3}}\left(\frac{3X+DY}{D+3}\right) \mp \sqrt{\frac{\gamma D}{D+2}}\left(\frac{X-Y}{D+3}\right)\right|}, \tag{70}$$

to make them square integrable. Now, following Hartle's point of view, we try to make correspondence between the classical loci and the wave functions.

Figures 1 - 6 show respectively the 2D plots of the typical wave functions Ψ_1^+ - Ψ_6^+ in terms of (X, Y) for $\gamma = 10^{-6}$; Figures 23 - 28 show the corresponding 3D plots, respectively. On the other hand, Figures 12 - 17 show the classical loci corresponding to $D = 1 - 6$, respectively. It is seen that the 2D and 3D plots of the wave functions Ψ_1^+ - Ψ_6^+ are exactly peaked on the classical loci.

In the same way, Figures 7 - 11 show respectively the 2D plots of the wave functions Ψ_2^- - Ψ_6^-. Figures 29 - 33 show the corresponding 3D plots, respectively. Figures 18 - 22 show the classical loci for $D = 2 - 6$, respectively. Again, an exact correspondence is seen between the 2D and 3D plots of the wave functions Ψ_2^- - Ψ_6^- and the classical loci. This procedure will apply for all D.

Concluding remarks

First, we have studied a $(4+D)$-dimensional classical Kaluza-Klein cosmology with a Robertson-Walker type metric having two scale factors, R for the universe and a for the higher dimensional space. By introducing a typical exotic matter with the equation of state $p_\chi = (\frac{m}{3} - 1)\rho_\chi$ in 4-dimensions, a decaying cosmological term is obtained effectively as $\lambda \sim R^{-m}$. By taking $m = 2$, the corresponding Einstein field equations are obtained and we find exponential solutions for R and a in terms of the Hubble parameter H. These exponential solutions indicate the accelerating expansion of the universe and dynamical compactification of extra dimensions, respectively. It turns out that the rate of compactification of extra dimensions as well as expansion of the universe depends on the number of extra dimensions, D. The more extra dimensions, the less rate of compactification and the more rate of acceleration. It is worth noting that the model is free of initial singularity problem because both R and a are non-zero at $t = 0$, resulting in a finite Ricci scalar.

Although the model describes in principle a closed universe with non-vanishing cosmological constant, it is equivalent to a flat universe with zero cosmological constant. Therefore, one may assume that we are really living in a closed universe with $\Lambda \neq 0$, but it effectively appears as a flat universe with $\Lambda = 0$. Note that we have not considered ordinary matter sources in the model except an exotic matter source which is to be considered as a source of dark energy. Therefore, it seems the solutions to describe typical inflation rather than the recently observed acceleration of the universe which is known to take place in an ordinary matter dominated universe. However, if the large percent of the matter sources in the universe would be of dark energy type (as the present observations strongly recommend), then one may keep the results here even in the presence of other matter source, keeping in mind that the relevant contribution to the total matter source of the universe is the dark energy.

A question may arise on the fact that no physics is supposed to exist below the planck length whereas for the contracting solution, the scale factor $a(t)$ goes to zero starting from l_p. However, it is not a major problem because we have not considered elements of quantum gravity theory in this model and merely studied a model based on general relativity which is supposed to be valid in any scale without limitation. The scale l_p, in this paper, is not introduced within a quantum gravity model (action); it just appears as a typical initial condition, in the middle of a classical model, based on the quantum cosmological consideration. One may choose another scale based on some other physical considerations.

We have also studied the corresponding quantum cosmology, through the Wheeler-DeWitt equation, and obtained the exact solutions. Based on Hartle'point of view on the correspondence between the classical and quantum solutions, we have shown by 2D and 3D plots of the wave functions a good correspondence between the classical and quantum cosmological solutions for any D, provided that the wave functions vanish for the infinite scale factors. There is no such a correspondence if another boundary condition, other than stated, is taken. Therefore, this correspondence guaranties that the chosen boundary condition is a *good* one.

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Figure captions

FIG. 1. 2D plot of Ψ_1^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 2. 2D plot of Ψ_2^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 3. 2D plot of Ψ_3^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 4. 2D plot of Ψ_4^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 5. 2D plot of Ψ_5^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 6. 2D plot of Ψ_6^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 7. 2D plot of Ψ_2^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 8. 2D plot of Ψ_3^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 9. 2D plot of Ψ_4^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 10. 2D plot of Ψ_5^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 11. 2D plot of Ψ_6^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 12. Classical locus $X + Y = 0$ for $D = 1$
FIG. 13. Classical locus $Y_- = \frac{2}{D}X[-1 - \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 2$
FIG. 14. Classical locus $Y_- = \frac{2}{D}X[-1 - \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 3$
FIG. 15. Classical locus $Y_- = \frac{2}{D}X[-1 - \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 4$
FIG. 16. Classical locus $Y_- = \frac{2}{D}X[-1 - \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 5$
FIG. 17. Classical locus $Y_- = \frac{2}{D}X[-1 - \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 6$
FIG. 18. Classical locus $Y_+ = \frac{2}{D}X[-1 + \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 2$
FIG. 19. Classical locus $Y_+ = \frac{2}{D}X[-1 + \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 3$
FIG. 20. Classical locus $Y_+ = \frac{2}{D}X[-1 + \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 4$
FIG. 21. Classical locus $Y_+ = \frac{2}{D}X[-1 + \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 5$
FIG. 22. Classical locus $Y_+ = \frac{2}{D}X[-1 + \sqrt{1 - \frac{2}{3}(1 - \frac{1}{D})}]^{-1}$ for $D = 6$
FIG. 23. 3D plot of Ψ_1^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 24. 3D plot of Ψ_2^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 25. 3D plot of Ψ_3^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 26. 3D plot of Ψ_4^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 27. 3D plot of Ψ_5^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 28. 3D plot of Ψ_6^+ in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 29. 3D plot of Ψ_2^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 30. 3D plot of Ψ_3^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 31. 3D plot of Ψ_4^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 32. 3D plot of Ψ_5^- in terms of (X, Y) for $\gamma = 10^{-6}$
FIG. 33. 3D plot of Ψ_6^- in terms of (X, Y) for $\gamma = 10^{-6}$

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